Turnitin_A Sequence of 1-Dimentional Subspace in a Normed Space

by Manuharawati Manuharawati

Submission date: 23-Sep-2019 09:12AM (UTC+0700)

Submission ID: 1177845492

File name: Manuharawati_2018_J._Phys._3A_Conf._Ser._1108_012085.pdf (494.25K)

Word count: 3389

Character count: 15350

PAPER • OPEN ACCESS

A Sequence Convergence of 1 –Dimensional Subspace in a Normed Space

To cite this article: Manuharawati *et al* 2018 *J. Phys.: Conf. Ser.* **1108** 012085

IOP Conf. Series: Journal of Physics: Conf. Series **1108** (2018) 012085 doi:10.1088/1742-6596/1108/1/012085

A Sequence Convergence of $1-$Dimensional Subspace in a Normed Space

Manuharawati[1,*], D N Yunianti[2], and M Jakfar[3]

[1, 2, 3] Department of Mathematics, Faculty of Mathematics and Natural Science, Universitas Negeri Surabaya, Indonesia
[*]Corresponding author: manuhara1@yahoo.co.id

Abstract. In this paper, the researchers will be introduced the concept of a sequence convergence of $1-$dimensional subspaces (lines) in a normed space and shall discuss some properties of it. Furthermore, it will be proved a continuity property of angles among subspaces in inner product spaces. Finally, the notion of limit of a sequence of $2-$dimensional subspaces (planes) in a normed space is studied. The researchers also obtain a result which describe how the convergent of a sequence of lines is associated to the convergent of a sequence of planes in a normed space.

1. Introduction

A sequence in the metric space and its properties has been widely used in investigating functional continuity [1], discussing fixed point theorems [2], discussing the central limit theorem in statistics [3], and discussing the numerical regression convergence [4].

The sequence in the metric space can be vector sequence, function sequence, matrix sequence, real numbers sequence, and so on. This has been discussed in some research conducted by many researchers:[5-13] discussing the convergence of vector sequences and their properties.

The application of vector sequences' convergence are widely studied by [14] [15]. The two research have discussed the angle between vectors in the inner product space and the concept of convergence of vector sequences in the inner product space are used to prove the continuous properties of the angular function between two vectors in the inner product space. In addition to both research, there are still many mathematicians who discuss the angle between subspace and its properties in the inner product space such as [3] [16] and [17]. However, neither of which shows the classification of the continuous properties of the angular function between the two subspaces in the inner product space.

The question arises: "Can we show the continuity of the angular function between two vector subspaces in the inner product space?" This will be answered if we have the convergence concept of subspace sequences from vector space. Readers may also agree if the concept is researched. It will also be obtained many applied to be produced.

In this paper, the researchers will define the convergence of sequence inone dimensional subspaces. This is a new concept introduced in the discussion of the convergence of sequence. The researchers also will discuss the basic properties of it. As a consequence of this definition, the property of continuity from the angle between the two subspaces will be shown. At the end, we will generalize the concept of convergence these subspaces in two-dimensional subspaces.

Before discussing the convergence concept of vector subspaces sequences, we first recall the definition of convergence of vector sequences as follows:

"a vector sequence (u_n) in normed space is said convergent to a vector u if $(\|u_n - u\|) \to 0$".

The definition of vector sequences convergence cannot be derived directly to define the convergence of subspace sequences. For example, a one-dimensional subspace can not only be replaced byu_n and u respectively as the basis of its subspace sequence, because when the definition is used, the convergence concept of subspaces sequences dependent on basis selection. Thus, by using the definition, there are subspaces sequences in normed space are convergent and divergent depending on the basis selection. For example $X = \mathbb{R}^2$, $U_n = span\{(1,0)\} = U \subset X$. If we take the basis of U_n where $\{u_n = (1,0)\}$for every n and the basis of U is $\{(1,0)\}$, then the sequence (U_n) converges to U because $(\|u_n - u\| = 0) \to 0$. But if we take the basis ofU_n is $\{u_n = (2,0)\}$ and $\{(1,0)\}$is the basis of U, then the sequence $(U_n)$is divergent to U because $\|u_n - u\| = \|(2,0) - (1,0)\| = \|(1,0)\| = 1 \nrightarrow 0$.

To overcome these weaknesses, it is necessary to construct the definition of convergent in finite dimensional subspace in normed space that independenton basis selection. According to the survey conducted by the author (journals, magazines, proceedings, national and international seminars), the concept of convergence in finite dimensional subspace in normed space has beenresearched by none.

If we choose $u$as the basis of one dimensional subspace $U = span\,\{u\}$, then for each $\alpha \in R$, αu is also the basis of U. Therefore, we can make the basis become the unit basis which the norm is 1 by multiplying the basis vector by $\frac{1}{||u||}$, or$\frac{u}{||u||}$. However, the one dimension subspace has two unit basis, that are$\frac{u}{||u||}$ and $\frac{-u}{||u||}$. Geometrically, it can be presented in Figure 1.

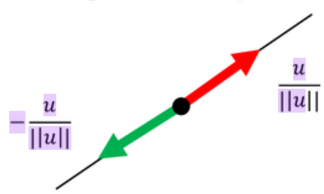

FIGURE 1. Representation of unit basis of a line.

By considering that the definition of $\|u_n - u\| \to 0$ for $n \to \infty$, it can be modified by

$$\left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\| \to 0$$

or

$$\left\|\frac{u_n}{\|u_n\|} + \frac{u}{\|u\|}\right\| \to 0$$

for $n \to \infty$.

However, if the result of modification is used as a definition, then the concept of convergence of line sequence is still dependent on thebasis selection.

For example, let $x = \mathbb{R}^2$, $U_n = span\{(1,0)\} = U \subset X$. If we take the base U_n for every n where $u_n = (1,0)$ and the basis of U is $(1,0)$, then sequences (HU_n) converges to U because $\left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\| = 0 \to 0$ for $n \to 0$. But if we take the basis of U_n is

$$u_n = \begin{cases}(1,0), & \text{if } n \text{ even}\\ (-1,0), & \text{if } n \text{ odd}\end{cases}$$

and the basis of U is $(1,0)$, then sequence (U_n) divergent to U, since both $\left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\|$ and $\left\|\frac{u_n}{\|u_n\|} + \frac{u}{\|u\|}\right\|$ divergent to 0 for $n \to 0$. So to resolve it will be modified again to be

$$\min\left\{\left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\|, \left\|\frac{u_n}{\|u_n\|} + \frac{u}{\|u\|}\right\|\right\} \to 0$$

for $n \to \infty$.

IOP Conf. Series: Journal of Physics: Conf. Series **1108** (2018) 012085 doi:10.1088/1742-6596/1108/1/012085

Next, Let X is a normed space. A 1-dimensional subspace of a normed space X is said a line in X, and 2-dimensional subspace of X is said a plane in X.

2. A Sequence of Lines

Let $\boldsymbol{X}$ is a normed space. Base on the introduction, we get the notion of limit of a sequence of lines in $\boldsymbol{X}$ (1-dimentional subspace of a normed space $\boldsymbol{X}$) and it will be the focus on this section.

Definition 2.1 Let (U_n) is a sequence of lines in X with $U_n = span\{u_n\}$ and $U = span\{u\}$ is a line in X. A sequence (U_n) is said to be **converge** to U, or U is is said to be a **limit** of (U_n), if and only if

$$\min\left\{\left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\|, \left\|\frac{u_n}{\|u_n\|} + \frac{u}{\|u\|}\right\|\right\} \to 0$$

as $n \to \infty$.

If a sequence has a limit, we say that the sequence is convergent; if it has no limit, we say that the sequence is divergent. We will sometimes use the symbolism $U_n \to U$, which indicates the intuitive idea that the lines U_n "approach" the line U as $n \to \infty$

The following theorem convinces us that our definition make sense.

Theorem 2.2. *The definition 2.1 satisfies homogeneity property. It means, The definition is dependent of the choice of bases for U and* U_n *for every* $n \in \mathbb{N}$.

Proof.

For every $n \in \mathbb{N}$, if u_n change with another basis of U_n, that is $\alpha_n u_n$ with $\alpha_n \in \mathbb{R}$ and u were replaced by βu with $\beta \in \mathbb{R}$, then there were two cases. If $\alpha_n \beta \geq 0$ then we find that

$$\left\|\frac{\alpha_n u_n}{\|\alpha_n u_n\|} - \frac{\beta u}{\|\beta u\|}\right\| = \left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\| \text{ and } \left\|\frac{\alpha_n u_n}{\|\alpha_n u_n\|} + \frac{\beta u}{\|\beta u\|}\right\| = \left\|\frac{u_n}{\|u_n\|} + \frac{u}{\|u\|}\right\|$$

On the other hand, if $\alpha_n \beta < 0$ then we get

$$\left\|\frac{\alpha_n u_n}{\|\alpha_n u_n\|} - \frac{\beta u}{\|\beta u\|}\right\| = \left\|\frac{u_n}{\|u_n\|} + \frac{u}{\|u\|}\right\| \text{ and } \left\|\frac{\alpha_n u_n}{\|\alpha_n u_n\|} + \frac{\beta u}{\|\beta u\|}\right\| = \left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\|.$$

Consequently,

$$\min\left\{\left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\|, \left\|\frac{u_n}{\|u_n\|} + \frac{u}{\|u\|}\right\|\right\} = \min\left\{\left\|\frac{\alpha_n u_n}{\|\alpha_n u_n\|} - \frac{\beta u}{\|\beta u\|}\right\|, \left\|\frac{\alpha_n u_n}{\|\alpha_n u_n\|} + \frac{\beta u}{\|\beta u\|}\right\|\right\}$$

■

Now, it will begin by establishing some basic properties of convergent sequences of lines since it will be needed in this section.

Fact 2.3. *If* $U_n = span\{u\} = U$ *then* $U_n \to U$.

Proof.

It is known that

$\left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\| = \left\|\frac{u}{\|u\|} - \frac{u}{\|u\|}\right\| = \|0\| = 0 \to 0$ as $n \to \infty$. Hence, $U_n \to U$. ■

Theorem 2.4. *Let* $U_n = span\{u_n\}$ *and* $U = span\{u\}$. *If* $U_n \to U$ *then*

$$\left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\| \left\|\frac{u_n}{\|u_n\|} + \frac{u}{\|u\|}\right\| \to 0$$

for $n \to \infty$.

Proof.

If $U_n \to U$ then, by Definition 2.1, it was obtained

$$\min\left\{\left\|\frac{u_n}{\|u_n\|}-\frac{u}{\|u\|}\right\|,\left\|\frac{u_n}{\|u_n\|}+\frac{u}{\|u\|}\right\|\right\}\to 0$$

as$n \to \infty$. Since $\left\|\frac{u_n}{\|u_n\|}+\frac{u}{\|u\|}\right\| \leq \left\|\frac{u_n}{\|u_n\|}\right\|+\left\|\frac{u}{\|u\|}\right\| = 2$ (Triangle Inequality) andalso $\left\|\frac{u_n}{\|u_n\|}-\frac{u}{\|u\|}\right\| \leq 2$, then

$$\left\|\frac{u_n}{\|u_n\|}-\frac{u}{\|u\|}\right\|\left\|\frac{u_n}{\|u_n\|}+\frac{u}{\|u\|}\right\| \leq 2\min\left\{\left\|\frac{u_n}{\|u_n\|}-\frac{u}{\|u\|}\right\|,\left\|\frac{u_n}{\|u_n\|}+\frac{u}{\|u\|}\right\|\right\}^2\to 0$$

as$n \to \infty$.■

Theorem 2.5. *Let$U_n = span\{u_n\}$ and $U = span\{u\}$. $U_n \to U$if and only if there exists $a_n \in \mathbb{R}, a_n \neq 0$ for every $n \in \mathbb{N}$such that*

$$\left\|\frac{u_n}{\|u_n\|}-\frac{a_n u}{\|a_n u\|}\right\|\to 0$$

as $n \to \infty$.

Proof.

($\Rightarrow$) If $U_n \to U$ then it was known that

$$\min\left\{\left\|\frac{u_n}{\|u_n\|}-\frac{u}{\|u\|}\right\|,\left\|\frac{u_n}{\|u_n\|}+\frac{u}{\|u\|}\right\|\right\}\to 0$$

as $n \to \infty$. So, There exists$a_n \in \mathbb{R}, a \neq 0$with$a_n \in \{1,-1\}$such that

$$\left\|\frac{u_n}{\|u_n\|}-\frac{au}{\|au\|}\right\|\to 0$$

as $n \to \infty$.

($\Leftarrow$) Let there exists$a_n \in \mathbb{R}, a_n \neq 0$ such that

$$\left\|\frac{u_n}{\|u_n\|}-\frac{a_n u}{\|a_n u\|}\right\|\to 0$$

as $n \to \infty$, then we have two cases, If$a_n > 0$ then it was obtained

$$\left\|\frac{u_n}{\|u_n\|}-\frac{u}{\|u\|}\right\| = \left\|\frac{u_n}{\|u_n\|}-\frac{a_n u}{\|a_n u\|}\right\| \text{ and } \left\|\frac{u_n}{\|u_n\|}+\frac{u}{\|u\|}\right\| = \left\|\frac{u_n}{\|u_n\|}+\frac{a_n u}{\|a_n u\|}\right\|,$$

if$a_n < 0$ then it was known that

$$\left\|\frac{u_n}{\|u_n\|}-\frac{u}{\|u\|}\right\| = \left\|\frac{u_n}{\|u_n\|}+\frac{a_n u}{\|a_n u\|}\right\| \text{ dan } \left\|\frac{u_n}{\|u_n\|}+\frac{u}{\|u\|}\right\| = \left\|\frac{u_n}{\|u_n\|}-\frac{a_n u}{\|a_n u\|}\right\|.$$

Finally,

$$\min\left\{\left\|\frac{u_n}{\|u_n\|}-\frac{u}{\|u\|}\right\|,\left\|\frac{u_n}{\|u_n\|}+\frac{u}{\|u\|}\right\|\right\} = \min\left\{\left\|\frac{u_n}{\|u_n\|}-\frac{a_n u}{\|a_n u\|}\right\|,\left\|\frac{u_n}{\|u_n\|}+\frac{a_n u}{\|a_n u\|}\right\|\right\}\to 0$$

as $n \to \infty$. So, $U_n \to \infty$. ■

Theorem 2.6. (Uniqueness of Limit) *Let$U_n = span\{u_n\}$ and $U = span\{u\}$. If $U_n \to U$and $U_n \to V$ then $U = V$.*

Proof.

Let $U_n \to U$. By applying Theorem 2.5, then we can find $a_n \in \mathbb{R}, a_n \neq 0$ for every $n \in \mathbb{N}$ such that

$$\left\|\frac{u_n}{\|u_n\|}-\frac{a_n u}{\|a_n u\|}\right\|\to 0$$

as $n \to \infty$. Since $U_n \to V$then according to Theorem 2.5, there exists $b_n \in \mathbb{R}, a_n \neq 0$ such that

$$\left\|\frac{u_n}{\|u_n\|}-\frac{b_n v}{\|b_n v\|}\right\|\to 0$$

as $n \to \infty$.

Next, by using Triangle Inequality, it was obtained that

$$\left\|\frac{a_n u}{\|a_n u\|}-\frac{b_n v}{\|b_n v\|}\right\| \leq \left\|\frac{u_n}{\|u_n\|}-\frac{a_n u}{\|a_n u\|}\right\|+\left\|\frac{u_n}{\|u_n\|}-\frac{b_n v}{\|b_n v\|}\right\|\to 0$$

as $n \to \infty$.

On the other hand, it was known that

IOP Conf. Series: Journal of Physics: Conf. Series **1108** (2018) 012085 doi:10.1088/1742-6596/1108/1/012085

$$\left\|\frac{a_nu_n}{\|a_nu_n\|}-\frac{b_nv}{\|b_nv\|}\right\|=\left\|\frac{a_nu_n}{|a_n|\|u_n\|}-\frac{b_nv}{|b_n|\|v\|}\right\|=\left\|\frac{u_n}{\|u_n\|}-\frac{|a_n|b_nv}{|b_n|a_n\|v\|}\right\|=\left\|\frac{u}{\|u\|}-\frac{(a_nb_n)v}{\|(a_nb_n)v\|}\right\|$$
$$=\left\|\frac{u}{\|u\|}\pm\frac{v}{\|v\|}\right\|.$$

Therefore, $\left\|\frac{u}{\|u\|}-\frac{v}{\|v\|}\right\|\to 0$or $\left\|\frac{u}{\|u\|}+\frac{v}{\|v\|}\right\|\to 0$. Hence, $\left\|\frac{u}{\|u\|}-\frac{v}{\|v\|}\right\|=0or\left\|\frac{u}{\|u\|}+\frac{v}{\|v\|}\right\|=0$. It means that $u=cv$ for some $c\in\mathbb{R}, c\neq 0$. So, $U=span\{u\}=span\{cv\}=span\{v\}=V$. ■

Theorem 2.7. *Let $U_n=span\{u_n\}$ and $U=span\{u\}$. $U_n\to U$if and only if for every $a\in\mathbb{R}, a_n\neq 0$, there exists $a\in\mathbb{R}, a_n\neq 0$ such that*
$$\left\|\frac{au_n}{\|au_n\|}-\frac{a_nu}{\|a_nu\|}\right\|\to 0$$
as $n\to\infty$.

Proof.

($\Rightarrow$) If $U_n\to U$ then by using Theorem 2.5, there were existed $a_n\in\mathbb{R}, a_n\neq 0$, such that
$$\left\|\frac{u_n}{\|u_n\|}-\frac{a_nu}{\|a_nu\|}\right\|\to 0$$
as $n\to\infty$. It meant that for all $a\in\mathbb{R}, a\neq 0$, if$a>0$ then
$$\left\|\frac{au_n}{\|au_n\|}-\frac{a_nu}{\|a_nu\|}\right\|=\left\|\frac{u_n}{\|u_n\|}-\frac{a_nu}{\|a_nu\|}\right\|\to 0,$$
If $a<0$, then there were exists $b_n\in\mathbb{R}, b_n\neq 0$, that is$b_n=-a_n$, such that
$$\left\|\frac{au_n}{\|au_n\|}-\frac{b_nu}{\|b_nu\|}\right\|=\left\|-\frac{u_n}{\|u_n\|}-\frac{-a_nu}{\|-a_nu\|}\right\|=\left\|\frac{u_n}{\|u_n\|}-\frac{a_nu}{\|a_nu\|}\right\|\to 0$$
as $n\to\infty$.

($\Leftarrow$) Let for every $a\in\mathbb{R}, a\neq 0$ there were exists $a_n\in\mathbb{R}, a_n\neq 0$ such that
$$\left\|\frac{au_n}{\|au_n\|}-\frac{a_nu}{\|a_nu\|}\right\|\to 0$$
as $n\to\infty$. There were two cases, if$a>0$ then it was obtained
$$\left\|\frac{u_n}{\|u_n\|}-\frac{a_nu}{\|a_nu\|}\right\|=\left\|\frac{au_n}{\|au_n\|}-\frac{a_nu}{\|a_nu\|}\right\|\to 0,$$
or if$a<0$ then it was obtained that
$$\left\|\frac{u_n}{\|u_n\|}-\frac{-a_nu}{\|-a_nu\|}\right\|=\left\|\frac{u_n}{\|u_n\|}+\frac{a_nu}{\|a_nu\|}\right\|=\left\|\frac{au_n}{\|au_n\|}-\frac{a_nu}{\|a_nu\|}\right\|\to 0,$$
as $n\to\infty$. So, by applying Theorem 2.5, $U_n\to U$. ■

3. Relationship to Angle in an Inner Product Space

Let $(X,\langle\cdot,\cdot\rangle)$ is an inner product spaces. Inspired by the notion of limit of a sequence of lines in X, it could be described the continuity property of angle between two lines.

Theorem 3.1. (Continuity Property) *Let $U_n, V_n, U$and Vare1-dimentional subspaces of a normed space $X$and $\theta(U,V)$ is an angle between $U$and V. If $U_n\to U$ and $V_n\to V$then $\theta(U_n,V_n)\to\theta(U,V)$.*

Proof.

Since $U_n\to U$ and $V_n\to V$ then by applying Theorem 2.5, it was obtained and found $a_n,b_n\in\mathbb{R}$, $a_n\neq 0$, and$b_n\neq 0$ for every $n\in\mathbb{N}$ such that
$$\left\|\frac{u_n}{\|u_n\|}-\frac{a_nu}{\|a_nu\|}\right\|\to 0\text{and}\left\|\frac{b_nv}{\|b_nv\|}-\frac{v_n}{\|v_n\|}\right\|\to 0 \text{ as } n\to\infty.$$
It was known that
$$\left|\left\|\frac{u_n}{\|u_n\|}-\frac{v_n}{\|v_n\|}\right\|-\left\|\frac{a_nu}{\|a_nu\|}-\frac{b_nv}{\|b_nv\|}\right\|\right|\leq\left\|\frac{u_n}{\|u_n\|}-\frac{a_nu}{\|a_nu\|}+\frac{b_nv}{\|b_nv\|}-\frac{v_n}{\|v_n\|}\right\|$$
$$\leq\left\|\frac{u_n}{\|u_n\|}-\frac{a_nu}{\|a_nu\|}\right\|+\left\|\frac{b_nv}{\|b_nv\|}-\frac{v_n}{\|v_n\|}\right\|\to 0$$

as $n \to \infty$. Therefore, $\left\|\frac{u_n}{\|u_n\|} - \frac{v_n}{\|v_n\|}\right\| \to \left\|\frac{a_n u}{\|a_n u\|} - \frac{b_n v}{\|b_n v\|}\right\|$ as $n \to \infty$. Hence, by the definition of angle between two subspaces of a normed space ([17]), it was obtained that

$$\theta(U_n, V_n) = \arccos\left|\frac{\langle u_n, v_n\rangle}{\|u_n\|\|v_n\|}\right| = \arccos\left|\frac{1}{2}\left(2 - \left\|\frac{u_n}{\|u_n\|} - \frac{v_n}{\|v_n\|}\right\|^2\right)\right|$$
$$\to \arccos\left|\frac{1}{2}\left(2 - \left\|\frac{a_n u}{\|a_n u\|} - \frac{b_n v}{\|b_n v\|}\right\|^2\right)\right| = \arccos\left|\frac{\langle a_n u, b_n v\rangle}{\|a_n u\|\|b_n v\|}\right|$$
$$= \arccos\left|\frac{\langle u, v\rangle}{\|u\|\|v\|}\right| = \theta(U, V)$$

as $n \to \infty$. ■

Relationship between the notion of limit of a sequence of lines in X and angle between vectors in X ([18] and [19]) is described in the following theorem.

Theorem 3.2. *Let $U_n = span\,\{u_n\}$ and $U = span\,\{u\}$ are 1-dimentional subspaces of a normed space X and $\theta_{Thy}(u, v)$ is an angle between two vector u and v. $U_n \to U$ if and only if $|\cos\theta_{Thy}(u_n, u)| \to 1$ as $n \to \infty$.*

Proof.
In [18] and [19], angle between u_n *and* u, $\theta_{Thy}(u_n, u)$ is defined as

$$\cos\theta_{Thy}(u_n, u) = \frac{1}{4}\left[\left\|\frac{u_n}{\|u_n\|} + \frac{u}{\|u\|}\right\|^2 - \left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\|^2\right].$$

In inner product spaces X, for $n \to \infty$, if $\left\|\frac{u_n}{\|u_n\|} + \frac{u}{\|u\|}\right\|^2 \to 0$ then $\left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\|^2 \to 4$, and if $\left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\|^2 \to 0$ then $\left\|\frac{u_n}{\|u_n\|} + \frac{u}{\|u\|}\right\|^2 \to 4$. Thus,

$$\min\left\{\left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\|, \left\|\frac{u_n}{\|u_n\|} + \frac{u}{\|u\|}\right\|\right\} \to 0 \leftrightarrow \frac{1}{4}\left[\left\|\frac{u_n}{\|u_n\|} + \frac{u}{\|u\|}\right\|^2 - \left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\|^2\right] \to 1. \;■$$

4. A Sequence of Planes

Theorem 2.7 was used to get the notion of limit of a sequence of planes in X (2-dimentional subspace of a normed space X).

Definition 4.1. Let (U_n) is a sequence of planes in X with $U_n = span\{u_{1_n}, u_{2_n}\}$ and $V = span\{v_1, v_2\}$is a plane in X. A sequence(U_n) is said to be converge to V if and only if for every, $b \in \mathbb{R}$, there exists $a_n, b_n \in \mathbb{R}$such that

$$\left\|\frac{(au_n + bv_n)}{\|au_n + bv_n\|} - \frac{(a_n u + b_n v)}{\|a_n u + b_n v\|}\right\| \to 0$$

As $n \to \infty$.

Note that $a, b \in \mathbb{R}$ in Definition 4.1 are arbitrary. It means that our definition is independent of the choice of basis for U_n and V. So, our definition make sense. Also, if $U_n$and V in Definition 4.1 are a 1-dimensional subspace of X, then the definition is equivalent with Theorem 2.7.

The following theorem describe a relation between the convergent of a sequence of lines in X and the convergent of a sequence of planes in X.

Theorem 4.2. Let $U_n = span\{u_n\}$, $V_n = span\{v_n\}$, $U = span\{u\}$, $V = span\{v\}$with u and v are linearly independent, and $B_n = span\{u_n, v_n\}$ and $B = span\{u, v\}$. If $U_n \to U$and $V_n \to V$ then $B_n \to B$.

Proof:
It was easy to prove this theorem by using Theorem 2.7, Definition 4.1, and Triangle Inequality.■

IOP Conf. Series: Journal of Physics: Conf. Series **1108** (2018) 012085 doi:10.1088/1742-6596/1108/1/012085

Remark 4.3. As the reader, it might have realized that the Definition 2.9 may also be generalized to define the limit of a sequence of m-dimentional subspace of a normed space. Let X is a normed space with $\dim X > m$. Let (V_n) is a sequence of m-dimentional subspaces of X with $V_n = span\{v_{1_n}, v_{2_n}, \dots, v_{m_n}\}$ and $V = span\{v_1, v_2, \dots, v_m\}$ is an m-dimentional subspace of X with$1_n, 2_n, \dots, m_n \in \mathbb{N}$. A sequence$(V_n)$ is said to be converge to V if and only if for every $a_1, a_2, \dots, a_m \in \mathbb{R}$there exists$b_{1_n}, b_{2_n}, \dots, b_{3_n} \in \mathbb{R}$ such that

$$\left\| \frac{\left(\sum_{i=1}^{m} a_i u_{i_n}\right)}{\left\|\sum_{i=1}^{m} a_i u_{i_n}\right\|} - \frac{\left(\sum_{j=1}^{m} b_{j_n} v_j\right)}{\left\|\sum_{j=1}^{m} b_{j_n} v_j\right\|} \right\| \to 0$$

as $n \to \infty$.

Note that $a_1, a_2, \dots, a_m$ in Definition 4.3 were arbitrary. It meant that the definition was also invariant under any change of basis for V_n and V. So, the definition made sense.

5. References

[1] Kreyszig E 1978 Introductory Functoinal Analysis with Applications (New York: John Wiley and Son)Komlos J 1972 *Every Sequence Converging to O Weakly in L_2 Contain An Unconditional Convergence Sequence* (Budapes: Publishing House of The Hunggarian Academi of Sci.)

[2] Davey B A and Priestly H A 2002 *Introduction to Lattices and Order* (Cambridge: Cambridge University Press)

[3] Voit J 2003 The Statistical Mechanics of Financial Markets *Springer-Verlag* **124**

[4] Isakov V 2006 SciencesInverse Problems for Partial Differential Equations (Second Edition *American Institute of Mathematical***127**

[5] Altman M 1957 A Fixed Point Theorem in Bannach Space*Bul. Acad. Polon. Sci.n Cl. III.***5** 19-22

[6] Alexits 1961 *Convergence Problems of Orthogonal Series* (Budapes: Publishing House of The Hunggarian Academi of Sci.)

[7] Powell 1976 Some Global convergence Properties of a Variable Metric Algorithm for Minimization without Exact Line Searches*SIAM-AMS Proceeding* **9**

[8] Parashar S D and Choudhary B 1994 Sequence Spaces Defined By Orlicz Functions*Indian J. Pure Appl. Math***25** (4) 419-428

[9] Quarteroni A, Sacco R, and Saleri F 2007 *Numerical Mathematics* (New York:Springer Berlin Heidelberg)

[10] Manuharawati and Yunianti D 2013a *Integral Henstock di dalam Ruang Metrik Kompak Lokal dari Fungsi Bernilai Vektor (*Surabaya: Laporan Penelitian, LPPM Unesa Surabaya).

[11] Manuharawati and Yunianti D 2013b Kriteria Chaucy dari Suatu Fungsi bernilai Vektor Pada Sel di Dalam Ruang Metrik Kompak Lokal*Prosiding Seminar Nasional Matematika dan Aplikasinya, Unair*

[12] Jakfar M 2018 Ruang Fungsi Terintegralkan p Berbobot *kubik* **3**(1) 60-67

[13] Jakfar M, Gunawan H, and Idris M 2015 *Hasil Kali Dalam Berbobot pada Ruang $L^p(X)$, Prosiding Semnastika 2015 Seminar Nasional Matematika dan Pendidikan Matematika* 298-304 (Surabaya: Unipress Unesa)

[14] Diminnie C R, Andalafte E Z and Freese R W 1988 Generalized Angles and a Characterization of Inner Product Spaces *Houston J. Math.***14** 475-480

[15] Gunawan H, Lindiarni J, Neswan O 2008 P-, I-, g-, and D-Angles in Normed Spaces*ITB J. Sci.***40 A** 24-32

[16] Risteski I B and Trenčevski K G 2001 Principal values and principal subspaces of two subspaces of vector spaces with inner product *Beitr. Algebra Geom* **42** 289–300

[17] Gunawan H, Neswan O and Setya-Budhi W 2005 A Formula for Angles between Subspaces of Inner Product Spaces*Beitr¨age zur Algebra und Geometrie Contributions to Algebra and Geometry***46** 311-320

[18] Thürey V 2009 Angles and polar coordinates in real normed spaces *arXiv:*0902.2731v2.

[19] Manuharawati and Jakfar M 2017 Thy-angle in an n-Normed *Space Far East Journal of Mathematical Sciences (FJMS)* **102**(5) 979-994

Turnitin_A Sequence of 1-Dimentional Subspace in a Normed Space

ORIGINALITY REPORT

13%	5%	12%	%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

1	Sergei Ovchinnikov. "Functional Analysis", Springer Nature America, Inc, 2018 Publication	2%
2	documents.mx Internet Source	1%
3	Thomas Borrvall. "Topology optimization of fluids in Stokes flow", International Journal for Numerical Methods in Fluids, 01/10/2003 Publication	1%
4	www.emis.de Internet Source	1%
5	www.media.umcom.org Internet Source	1%
6	Elements of Operator Theory, 2001. Publication	1%
7	Gyorgy Hexner. "LQG Guidance Law with Bounded Acceleration Command", IEEE Transactions on Aerospace and Electronic	1%

Systems, 1/2008

Publication

8	Hendra Gunawan. "n-Inner Products, n-Norms, and Angles Between Two Subspaces", Wiley, 2018 Publication	1%
9	MATÍAS MENNI, ALEX SIMPSON. "Topological and limit-space subcategories of countably-based equilogical spaces", Mathematical Structures in Computer Science, 2003 Publication	1%
10	Xuan, B.. "The solvability of quasilinear Brezis-Nirenberg-type problems with singular weights", Nonlinear Analysis, 20050815 Publication	1%
11	José M. Gallardo. "A generalized duality method for solving variational inequalities Applications to some nonlinear Dirichlet problems", Numerische Mathematik, 04/2005 Publication	1%
12	calculus7.org Internet Source	<1%
13	epdf.pub Internet Source	<1%
14	Darpo, E.. "Normal forms for the G"2-action on the real symmetric 7x7-matrices by	<1%

conjugation", Journal of Algebra, 20070615

Publication

15	emis.library.cornell.edu Internet Source	<1%
16	M. Nur, H. Gunawan, O. Neswan. "A formula for the g-angle between two subspaces of a normed space", Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry, 2017 Publication	<1%
17	"Approximation Theory", Texts in Applied Mathematics, 2005 Publication	<1%
18	Advanced Information and Knowledge Processing, 2014. Publication	<1%
19	O. Besson. "Hele-Shaw approximation for resin transfer molding", ZAMM, 04/13/2005 Publication	<1%
20	Vakeel A. Khan, Kamal M.A.S. Alshlool, Sameera A.A. Abdullah, Rami K.A. Rababah, Ayaz Ahmad. "Some new classes of paranorm ideal convergent double sequences of sigma-bounded variation over n-normed spaces", Cogent Mathematics & Statistics, 2018 Publication	<1%

21	epubs.siam.org Internet Source	<1%
22	Hongxia Liu. "Interaction of elementary waves for scalar conservation laws on a bounded domain", Mathematical Methods in the Applied Sciences, 05/10/2003 Publication	<1%
23	M. Nur, H. Gunawan. "A new orthogonality and angle in a normed space", Aequationes mathematicae, 2018 Publication	<1%
24	Ricardo, . "Inner Product Spaces", A Modern Introduction to Linear Algebra, 2009. Publication	<1%
25	Masahiro Okuno-Fujiwara. "Monitoring cost, agency relationships, and equilibrium modes of labor contracts", Journal of the Japanese and International Economies, 1987 Publication	<1%

Exclude quotes	Off	Exclude matches	< 3 words
Exclude bibliography	On		